CERTAIN MULTI(SUB)LINEAR SQUARE FUNCTIONS

LOUKAS GRAFAKOS, SHA HE, AND QINGYING XUE

Abstract. Let $d \geq 1, \ell \in \mathbb{Z}^d$, $m \in \mathbb{Z}^+$ and θ_i, $i = 1, \dots, m$ are fixed, distinct and nonzero real numbers. We show that the m-(sub)linear version below of the Ratnakumar and Shrivastava [11] Littlewood-Paley square function

$$T(f_1, \dots, f_m)(x) = \Big(\sum_{\ell \in \mathbb{Z}^d} \big| \int_{\mathbb{R}^d} f_1(x - \theta_1 y) \cdots f_m(x - \theta_m y) e^{2\pi i \ell \cdot y} K(y) dy \big|^2 \Big)^{1/2}$$

is bounded from $L^{p_1}(\mathbb{R}^d) \times \cdots \times L^{p_m}(\mathbb{R}^d)$ to $L^p(\mathbb{R}^d)$ when $2 \leq p_i < \infty$ satisfy $1/p = 1/p_1 + \cdots + 1/p_m$ and $1 \leq p < \infty$. Our proof is based on a modification of an inequality of Guliyev and Nazirova [6] concerning multilinear convolutions.

1. Introduction and main results

Motivated by the study of the bilinear Hilbert transform

$$H(f, g)(x) = p.v. \frac{1}{\pi} \int_{\mathbb{R}} f(x - y) g(x + y) \frac{dy}{y}, \tag{1.1}$$

by Lacey and Thiele [8], [9], a substantial amount of work has been produced in the area of multilinear singular integral and multiplier operators. In this paper we study certain kinds of multi(sub)linear square functions.

We introduce a bilinear operator which is closely related to the bilinear Hilbert transform. Let w be a cube in $\mathbb{R}^d$, $d \geq 1$. Then for functions f, g in the Schwartz class $\mathcal{S}(\mathbb{R}^d)$, we define the bilinear operator S_w associated with the symbol $\chi_w(\xi - \eta)$ as follows:

$$S_w(f, g)(x) = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \hat{f}(\xi) \hat{g}(\eta) \chi_w(\xi - \eta) e^{2\pi i x \cdot (\xi + \eta)} d\xi d\eta.$$

We notice that when $d = 1$ and w is the characteristic function of a half plane in $\mathbb{R}^2$, then S_w is related to the bilinear Hilbert transform in (1.1).

Let $\{w_l\}_{l \in \mathbb{Z}^d}$ be a sequence of disjoint cubes in $\mathbb{R}^d$. Let S_{w_l} be the bilinear operator associated with the symbol $\chi_{w_l}(\xi - \eta)$ as defined above. Then for $f, g \in \mathcal{S}(\mathbb{R}^d)$, the *non-smooth bilinear Littlewood-Paley square function* associated with the sequence $\{w_l\}_{l \in \mathbb{Z}^d}$ is defined as

$$S(f, g)(x) = \Big(\sum_{l \in \mathbb{Z}^d} |S_{w_l}(f, g)(x)|^2 \Big)^{1/2}. \tag{1.2}$$

2000 *Mathematics Subject Classification.* Primary 42B20; Secondary 42B25.

Key words and phrases. Multilinear operators, Littlewood-Paley square functions, orthogonality.

The first author was supported partly by the Simons Foundation. The third author was supported partly by NSFC (No. 11471041), the Fundamental Research Funds for the Central Universities (No. 2014KJJCA10) and NCET-13-0065.

A *smooth* version of the bilinear Littlewood-Paley square function in (1.2) can be obtained if the characteristic function of the cube w is replaced by a smooth bump adapted to w. Precisely, we let

$$T(f,g)(x) = \Big(\sum_{l\in\mathbb{Z}^d} |T_{\phi_l}(f,g)(x)|^2 \Big)^{1/2}, \tag{1.3}$$

where T_{ϕ_l} is a bilinear operator associated with the smooth function ϕ_l whose Fourier transform is supported in w_l, i.e.,

$$T_{\phi_l}(f,g)(x) = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \widehat{f}(\xi)\widehat{g}(\eta)\widehat{\phi_l}(\xi-\eta)e^{2\pi i x\cdot(\xi+\eta)}d\xi d\eta.$$

A particular case of the bilinear Littlewood-Paley square function T in (1.3) was studied by Lacey [7] who proved the following:

Theorem A ([7]). Let ϕ be a smooth function on $\mathbb{R}^d$ such that $\hat{\phi}$ is supported in the unit cube of $\mathbb{R}^d$. For $l \in \mathbb{Z}^d$, let $\widehat{\phi_l}$ be the function defined by $\widehat{\phi_l}(\xi) = \widehat{\phi}(\xi - l)$. Then for $2 \le p, q \le \infty$ with $1/p + 1/q = 1/2$, there is a constant C such that for $f, g \in \mathcal{S}(\mathbb{R}^d)$,

$$\|T(f,g)\|_{L^2(\mathbb{R}^d)} \le C\|f\|_{L^p(\mathbb{R}^d)}\|g\|_{L^q(\mathbb{R}^d)}.$$

In addition, Lacey raised two questions related to this: (a) Does Theorem A hold for $r \neq 2$? (b) Are there analogues for non-smooth square functions? The boundedness of non-smooth square functions on $\mathbb{R}\times\mathbb{R}$ strengthens the boundedness of bilinear Hilbert transform; in this direction and in the spirit of question (b), Bernicot [2], Ratnakumar and Shrivastava [12] have provided some interesting results. As far as question (a), answers were provided by Bernicot and Shrivastava [4], Mohanty and Shrivastava [10]. The proofs of these results were based on rather complicated time-frequency analysis. However, Ratnakumar and Shrivastava [11] provided a proof for the boundedness of a smooth bilinear square functions, which is based on more elementary techniques. Motivated by the work of [11] and the increasing interest in multilinear operators, the aim of this paper is to study the L^p boundedness properties of smooth m-(sub)linear Littlewood-Paley square functions.

Definition 1.1. We define an *m-(sub)linear Littlewood-Paley square function* as:

$$T(f_1,\dots,f_m)(x) = \Big(\sum_{\ell\in\mathbb{Z}^d} |T_\ell(f_1,\dots,f_m)(x)|^2 \Big)^{1/2}, \tag{1.4}$$

where

$$T_\ell(f_1,\dots,f_m)(x) = \int_{\mathbb{R}^d} f_1(x-\theta_1 y)\cdots f_m(x-\theta_m y)K_\ell(y)dy. \tag{1.5}$$

Here θ_j, $j = 1,\dots,m$ are fixed, distinct and nonzero real numbers, and K_ℓ are integrable functions that satisfy certain additional size conditions.

Expressing each T_ℓ in m-linear Fourier multiplier form, we write

$$T_\ell(f_1,\dots,f_m)(x) = \int_{(\mathbb{R}^d)^m} \Big[\prod_{i=1}^m (\widehat{f_i}(\xi_i)e^{2\pi i x\cdot\xi_i})\Big]\widehat{K_\ell}(\theta_1\xi_1+\cdots+\theta_m\xi_m)d\xi_1\cdots d\xi_m.$$

Then T_ℓ is associated with the multiplier (or symbol) $\widehat{K_\ell}(\theta_1\xi_1 + \cdots + \theta_m\xi_m)$.

We use the notation $r' = r/(r-1)$ for the dual exponent of $r \in [1,\infty]$ with $1' = \infty$ and $\infty' = 1$. Our main result is the following:

Theorem 1.1. *Let $1 \le p < \infty$ and let K be a measurable function on $\mathbb{R}^d$ which satisfies*

$$B_p = \sum_{u\in\mathbb{Z}^d} \big\|\chi_{Q_u} K\big\|_{L^{p'}(\mathbb{R}^d)} < \infty \qquad \text{when } 1 \le p < 2 \tag{1.6}$$

and

$$B_2 = \sum_{u\in\mathbb{Z}^d} \big\|\chi_{Q_u} K\big\|_{L^{2}(\mathbb{R}^d)} < \infty \qquad \text{when } 2 \le p < \infty,$$

where $Q_u = u + [0,1)^d$ for $u \in \mathbb{Z}^d$. For $\ell \in \mathbb{Z}^d$ define $K_\ell(x) = K(x)e^{2\pi i x\cdot\ell}$ and let T_ℓ be the m-linear Fourier multiplier in (1.5), where the θ_j are nonzero and distinct. Then for $2 \le p_j < \infty$, $j = 1,\dots,m$, there exists a positive constant $C = C(d,\theta_j,p_j)$ such that the square function T in (1.4) satisfies

$$\|T(f_1,\dots,f_m)\|_{L^p(\mathbb{R}^d)} \le BC\prod_{j=1}^m \|f_j\|_{L^{p_j}(\mathbb{R}^d)} \tag{1.7}$$

for all functions f_j in $L^{p_j}(\mathbb{R}^d)$, where p and p_j are related via

$$\frac{1}{p} = \sum_{j=1}^m \frac{1}{p_j} \tag{1.8}$$

and B in (1.7) is either B_2 if $2 \le p < \infty$ or B_p when $1 \le p < 2$.

The case $m = 2$ of this theorem was obtained by Ratnakumar and Shrivastava [11]. The new ingredient of this note is the extension of this result to the case where $m \ge 3$, where a multilinear version of Young's inequality is needed. Such an inequality was obtained by Guliyev and Nazirova [6] for a certain range of exponents which is not sufficient for our work and for this reason, we provide an extension of this paper. To state the multilinear Young inequality of Guliyev and Nazirova, we fix θ_i nonzero distinct real numbers and we define

$$\vec{f}\otimes g(x) = \int_{\mathbb{R}^d} f_1(x-\theta_1 y)\cdots f_m(x-\theta_m y)g(y)dy,$$

where $\vec{f} = (f_1,\dots,f_m)$ and f_j, g are measurable functions so that the preceding integral converges. We denote by $L^{p,\infty}(\mathbb{R}^d)$ the weak L^p space of all measurable functions f such that $\|f\|_{L^{p,\infty}} = \sup_{\alpha>0}\big\{\alpha|\{x\in\mathbb{R}^d : |f(x)| > \alpha\}|^{1/p}\big\} < \infty$.

Theorem 1.2. *([6]) Assume that $1 < r < \infty$, $1 < p_j \le \infty$, $i = 1,\dots,m$, $1/p = 1/p_1 + \cdots + 1/p_m$, $1/q + 1 = 1/p + 1/r$, and $\frac{r'}{1+r'} \le p < r'$ (equivalently $1 \le q < \infty$). Then for $g \in L^{r,\infty}(\mathbb{R}^d)$ and $f_j \in L^{p_j}(\mathbb{R}^d)$ we have $\vec{f}\otimes g \in L^q(\mathbb{R}^d)$ with norm inequality*

$$\|\vec{f}\otimes g\|_{L^q(\mathbb{R}^d)} \le C(d,\theta_j,p_j,r)\prod_{i=1}^m \|f_i\|_{L^{p_i}(\mathbb{R}^d)}\|g\|_{L^{r,\infty}(\mathbb{R}^d)}.$$

Unfortunately, the proof of Theorem 1.2 in [6] in the range $\frac{r'}{r'+1} \le p \le 1$ (equivalently $1 \le q \le r$) contains an inaccurate deduction (and the assertion is incorrect in this case). In the proof of Theorem 1.1 we need the case where $\frac{r'}{1+r'} = p$ (equivalently $q = 1$) and for this reason we provide an analogue in which the weak norm of g is replaced by a strong norm. So we fix the inaccurate deduction in [6] via the following result:

Theorem 1.3. *Assume that $1 < r \le \infty$, $1/p = 1/p_1 + \cdots + 1/p_m$, $1 \le p_j \le \infty$, $j = 1, \dots, m$, $1/q + 1 = 1/p + 1/r$, and $\frac{r'}{1+r'} \le p \le r'$ (equivalently $1 \le q \le \infty$). Then for some constant $C(d, \theta_j, p_j, r)$ the following inequalities are valid:*
(a) if $1 < r < \infty$, then when $1 < p < r'$ (equivalently $r < q < \infty$) we have

$$\|\vec{f} \otimes g\|_{L^q(\mathbb{R}^d)} \le C(d, \theta_j, p_j, r) \prod_{i=1}^{m} \|f_i\|_{L^{p_i}(\mathbb{R}^d)} \|g\|_{L^{r,\infty}(\mathbb{R}^d)}.$$

(b) if $1 < r \le \infty$, then when $\frac{r'}{1+r'} \le p \le 1$ (equivalently $1 \le q \le r$) we have

$$\|\vec{f} \otimes g\|_{L^q(\mathbb{R}^d)} \le C(d, \theta_j, p_j, r) \prod_{i=1}^{m} \|f_i\|_{L^{p_i}(\mathbb{R}^d)} \|g\|_{L^{r}(\mathbb{R}^d)}.$$

Remark 1.2. We notice that in Theorem 1.3 we are assuming that $p_j \ge 1$ for all j whereas in [6] it is assumed that $p_j > 1$ for all j. This small detail, i.e., the case $p_j = 1$ for some j, does not appear in case (a), since we always have $p \le p_j$ for all j and in case (a) we have $p > 1$. Thus the case where some p_j equals 1 is relevant only in case (b), which is treated in the next section.

2. The proof of Theorem 1.3

The case where $\frac{r'}{r'+1} \le p \le 1$, i.e., case (a) of Theorem 1.3 is contained in [6]. So we focus attention in case (b) of Theorem 1.3, i.e., when $\frac{r'}{1+r'} \le p \le r'$ (equivalently $1 \le q \le r$). Below we assume that $1 < r \le \infty$.

First we consider the case $p = 1$ when $q = r$. Since $1/p_1 + \cdots + 1/p_m = 1$, making use of Hölder's inequality and Fubini's theorem, we write

$$\begin{aligned}
&\|\vec{f} \otimes g\|_{L^r(\mathbb{R}^d)} \\
&\le \Big(\int_{\mathbb{R}^d} \Big(\int_{\mathbb{R}^d} |f_1(x-\theta_1 y) \cdots f_m(x-\theta_m y) g(y)| dy \Big)^r dx \Big)^{\frac{1}{r}} \\
&\le \Big(\int_{\mathbb{R}^d} \Big\{ \Big(\int_{\mathbb{R}^d} |g(y)|^r |f_1(x-\theta_1 y) \cdots f_m(x-\theta_m y)| dy \Big)^{\frac{1}{r}} \\
&\qquad\qquad \Big(\int_{\mathbb{R}^d} |f_1(x-\theta_1 y) \cdots f_m(x-\theta_m y)| dy \Big)^{\frac{1}{r'}} \Big\}^r dx \Big)^{\frac{1}{r}} \\
&\le \Big(\prod_{i=1}^{m} |\theta_i|^{-\frac{d}{p_i r'}} \Big) \Big(\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |g(y)|^r |f_1(x-\theta_1 y) \cdots f_m(x-\theta_m y)| dy\, dx \Big)^{\frac{1}{r}} \prod_{i=1}^{m} \|f_i\|_{L^{p_i}}^{\frac{1}{r'}}
\end{aligned}$$

$$
\begin{aligned}
&= \Big(\prod_{i=1}^m |\theta_i|^{-\frac{d}{p_i r'}}\Big)\Big(\int_{\mathbb{R}^d} |g(y)|^r \int_{\mathbb{R}^d} |f_1(x-\theta_1 y)\cdots f_m(x-\theta_m y)|dx\, dy\Big)^{\frac{1}{r}} \prod_{i=1}^m \|f_i\|_{L^{p_i}}^{\frac{1}{r'}} \\
&\le \Big(\prod_{i=1}^m |\theta_i|^{-\frac{d}{p_i r'}}\Big)\|g\|_{L^r} \prod_{i=1}^m \|f_i\|_{L^{p_i}}^{\frac{1}{r}} \prod_{i=1}^m \|f_i\|_{L^{p_i}}^{\frac{1}{r'}} \\
&= \Big(\prod_{i=1}^m |\theta_i|^{-\frac{d}{p_i r'}}\Big)\|g\|_{L^r} \prod_{i=1}^m \|f_i\|_{L^{p_i}},
\end{aligned}
$$

where in the second inequality we apply Hölder's inequality with respect to the measure $\prod_{i=1}^m |f_i(x-\theta_i y)|dy$ to the function $y \mapsto |g(y)|$ and 1 with exponents r and r', respectively. Notice that this proof also works in the case where $r=\infty$ with a small notational modification.

Next we consider the case $p = \frac{r'}{r'+1}$, which is equivalent to $q=1$. In this case r could be equal to ∞ and thus r' could take the value 1 when $p=1/2$. Let us give the proof first in the case where $m=3$, where the notation is simpler. In this case when $r'>1$ notice that $1/p_1+1/p_2+1/p_3 = 1+1/r' < 2$, thus in the space with coordinates (x_1,x_2,x_3), the intersection of the plane $x_1+x_2+x_3 = 1+1/r'$ and of the boundary of the unit cube $[0,1]^3$ is a closed hexagon with vertices at six points $V_1^{(3)} = (1/r',0,1)$, $V_2^{(3)} = (1/r',1,0)$, $V_3^{(3)} = (1,0,1/r')$, $V_4^{(3)} = (0,1,1/r')$, $V_5^{(3)} = (1,1/r',0)$, $V_6^{(3)} = (0,1/r',1)$. In the case where $r=\infty$, i.e., $r'=1$, the intersection of boundary of the unit cube $[0,1]^3$ with the plane $x_1+x_2+x_3=2$ is a closed triangle with vertices the points $W_1^{(3)} = (0,1,1)$, $W_2^{(3)} = (1,0,1)$ and $W_3^{(3)} = (1,1,0)$.

We assert that $\vec{f}\otimes g$ maps $L^{p_1}\times L^{p_2}\times L^{p_3}\times L^r$ into L^1 when $(1/p_1,1/p_2,1/p_3)$ lies in the closed convex hull of the six points $V_j^{(3)}$ when $r<\infty$ (or the three points $W_1^{(3)}$, $W_2^{(3)}$, $W_3^{(3)}$ when $r=\infty$). This assertion will be a consequence of the fact that boundedness holds at the vertices via multilinear complex interpolation (in particular by applying the theorem in Zygmund [13, Chapter XII, (3.3)] or in Berg and Löfstrom [1, Theorem 4.4.2] or the main result in [5]).

We only prove boundedness for point $V_1^{(3)}$, since the method is similar for the remaining vertices. We have

$$
\begin{aligned}
&\|\vec{f}\otimes g\|_{L^1(\mathbb{R}^d)} \\
&\le \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} |f_1(x-\theta_1 y)||f_2(x-\theta_2 y)||f_3(x-\theta_3 y)||g(y)|dydx \\
&= \int_{\mathbb{R}^d} |g(y)| \int_{\mathbb{R}^d} |f_1(x-\theta_1 y)||f_2(x-\theta_2 y)||f_3(x-\theta_3 y)|dxdy \\
&\le \Big(\int_{\mathbb{R}^d} |g(y)|^r dy\Big)^{\frac{1}{r}} \Big(\int_{\mathbb{R}^d}\Big(\int_{\mathbb{R}^d} |f_1(x-\theta_1 y)||f_2(x-\theta_2 y)||f_3(x-\theta_3 y)|dx\Big)^{r'} dy\Big)^{\frac{1}{r'}} \\
&\le \|g\|_{L^r}\|f_2\|_{L^\infty}\Big(\int_{\mathbb{R}^d}\Big(\int_{\mathbb{R}^d} |f_1(t)||f_3(t-(\theta_3-\theta_2)y)|dt\Big)^{r'} dy\Big)^{\frac{1}{r'}}
\end{aligned}
$$

$$\leq |\theta_3 - \theta_2|^{-\frac{d}{r'}} \|g\|_{L^r} \|f_1\|_{L^{r'}} \|f_2\|_{L^\infty} \|f_3\|_{L^1},$$

where in the last inequality we use Young's inequality, in view of the fact that $p_1 = r'$ and $p_3 = 1$ for the point $V_1^{(3)}$, which implies $1/p_1 + 1/p_3 = 1 + 1/r'$. Note that r could be equal to infinity in this case. Boundedness for the remaining vertices follows by symmetry.

The case $m \geq 4$ is proved via the same method. Let $E^{(m)}$ be the intersection of the boundary of the cube $[0,1]^m$ in $\mathbb{R}^m$ with the hyperplane $x_1 + \cdots + x_m = 1 + 1/r'$. Let us first consider the case where $r' > 1$. The set $E^{(m)}$ is a closed convex polygon with $m(m-1)$ vertices which we call $V_i^{(m)}$. These are the points in $\mathbb{R}^m$ whose coordinates are all zero except for two of them which are 1 and $1/r'$. We claim that $E^{(m)}$ is the closed convex hull of the vertices $V_i^{(m)}$. In fact, we show this by induction, as follows: The assertion is true when $m = 3$. If this assertion is valid when $m = n - 1$ for some $n \geq 4$, then the faces of $E^{(n)}$ are the sets

$$\{(1/q_1, \dots, 1/q_n) : \sum_{i=1}^{n} q_i^{-1} = 1 + 1/r', \quad 1 \leq q_i \leq \infty, \text{ and } q_i = \infty \text{ for only one } i\}.$$

Consider for instance the face

$$F_1^{(n)} = \{(1/q_1, \dots, 1/q_n) : \sum_{i=1}^{n} q_i^{-1} = 1 + 1/r', \quad 1 \leq q_i \leq \infty \text{ and } q_1 = \infty\}.$$

Then $F_1^{(n)}$ can be identified with the set

$$\{(1/q_2, \dots, 1/q_n) : \sum_{i=2}^{n} q_i^{-1} = 1 + 1/r', \quad 1 \leq q_i \leq \infty\}$$

which is equal to the intersection of the cube $[0,1]^{m-1}$ with the hyperplane $\sum_{i=2}^n x_i = 1 + 1/r'$ in $\mathbb{R}^{m-1}$ and corresponds to the case $m = n - 1$. Applying the induction hypothesis, the vertices of $F_1^{(n)}$ are the points $(1/q_1, \dots, 1/q_n)$ where $1/q_1 = 0$ and the $1/q_i$ with $i \geq 2$ are zero except for two of them which are either 1 or $1/r'$. Thus, since $1/q_1 = 0$, the vertices of $F_1^{(n)}$ are the points $(1/q_1, \dots, 1/q_n)$ where $1/q_1 = 0$ and $1/q_i$ with $i \geq 2$ are zero except for two of them which are either 1 or $1/r'$. The same holds for the remaining faces and thus the assertion that the points $V_i^{(m)}$ are the vertices of $E^{(m)}$ holds when $m = n$. The same conclusion holds when $r' = 1$, but in this case the intersection of the hyperplane $x_1 + \cdots + x_m = 2$ with the cube $[0,1]^m$ is a closed convex set with $m(m-1)/2$ vertices $W_i^{(m)}$ instead of $m(m-1)$ vertices. Observe that the set of $W_i^{(m)}$ is the set of all vectors with in $\mathbb{R}^m$ whose coordinates are zero except for two of them which are equal to 1.

Notice that at each vertex $V_i^{(m)}$ (or $W_i^{(m)}$) boundedness holds and is proved exactly in the same way as when $m = 3$ with the only exception being that the L^∞ norm of one function is replaced by the product of L^∞ norms of the remaining $m-2$ functions. Then multilinear complex interpolation yields boundedness in the closed convex hull of these vertices, i.e., yields the required conclusion when $p = \frac{r'}{r'+1}$ (equivalently $q = 1$).

Finally, the case where $\frac{r'}{1+r'} < p < 1$ (equivalently $1 < q < r$) is obtained by interpolation between the endpoint cases where $p = \frac{r'}{r'+1}$ (equivalently $q = 1$) and $p = 1$ (equivalently $q = r$). Notice that in the case where $p = 1$ ($q = r$) the intersection of the boundary of the unit cube $[0,1]^m$ with the hyperplane $x_1 + \cdots + x_m = 1$ is the closed convex hull of the m points $U_j^{(m)} = (0, \ldots, 1, \ldots, 0)$ with 1 in the jth coordinate and zero in every other coordinate. Let $Z_k^{(m)}$ be the vertices of the closed convex set formed by the intersection of the boundary of the unit cube $[0,1]^m$ with the hyperplane $x_1 + \cdots + x_m = 1/q + 1/r'$ in $\mathbb{R}^m$ when $1 < q < r$. Then there is a $\theta \in (0,1)$ [in fact $\theta = r'(\frac{1}{q} - \frac{1}{r})$] such that the set of all $Z_k^{(m)}$ is contained in the set

$$\Big\{(1-\theta)U_j^{(m)} + \theta V_i^{(m)}: \qquad j \in \{1, \ldots, m\}, \quad i \in \{1, \ldots, m(m-1)\}\Big\},$$

where $V_i^{(m)}$ is replaced by $W_i^{(m)}$ for $i \in \{1, \ldots, m(m-1)/2\}$, if $r = \infty$. Thus interpolation is possible in this case.

3. Proof of Theorem 1.1

Proof. Using the inverse Fourier transform we write

$$\begin{aligned} T_\ell(f_1, \ldots, f_m)(x) &= \int_{\mathbb{R}^d} f_1(x - \theta_1 y) \cdots f_m(x - \theta_m y) K_\ell(y) dy \\ &= \int_{\mathbb{R}^d} f_1(x - \theta_1 y) \cdots f_m(x - \theta_m y) e^{2\pi i \ell \cdot y} K(y) dy. \end{aligned}$$

We claim that the following inequality holds for T:

$$\begin{aligned} T(f_1, \ldots, f_m)(x) &= \Big(\sum_{\ell \in \mathbb{Z}^d} |T_\ell(f_1, \ldots, f_m)(x)|^2\Big)^{\frac{1}{2}} \\ &\le \sum_{u \in \mathbb{Z}^d} \Big(\int_{Q_u} |f_1(x - \theta_1 y) \cdots f_m(x - \theta_m y) K(y)|^2 dy\Big)^{\frac{1}{2}}. \end{aligned} \tag{3.1}$$

To verify the validity of this claim, we take a sequence $a = \{a_\ell\}_{\ell \in \mathbb{Z}^d} \in \ell_2(\mathbb{Z}^d)$ with the property $\|a\|_{\ell_2(\mathbb{Z}^d)} = 1$. Then by duality, we need to prove that

$$\Big|\sum_{\ell \in \mathbb{Z}^d} a_\ell T_\ell(f_1, \ldots, f_m)(x)\Big| \le \sum_{u \in \mathbb{Z}^d} \Big(\int_{Q_u} |f_1(x - \theta_1 y) \cdots f_m(x - \theta_m y) K(y)|^2 dy\Big)^{\frac{1}{2}}.$$

Note that

$$\begin{aligned} \sum_{\ell \in \mathbb{Z}^d} a_\ell T_\ell(f_1, \ldots, f_m)(x) &= \int_{\mathbb{R}^d} f_1(x - \theta_1 y) \cdots f_m(x - \theta_m y) K(y) \sum_{\ell \in \mathbb{Z}^d} a_\ell e^{2\pi i \ell \cdot y} dy \\ &= \int_{\mathbb{R}^d} f_1(x - \theta_1 y) \cdots f_m(x - \theta_m y) K(y) \hat{a}(y) dy, \end{aligned}$$

where $\hat{a}$ is the Fourier transform of sequence a. It is easy to see that $\hat{a}$ is a periodic function with $\|\hat{a}\|_{L^2([0,1]^d)} = 1$. Hence,

$$\begin{aligned}
&\Big|\sum_{\ell\in\mathbb{Z}^d} a_\ell T_\ell(f_1,\dots,f_m)(x)\Big| \\
&\le \int_{\mathbb{R}^d} |f_1(x-\theta_1 y)\cdots f_m(x-\theta_m y)K(y)\hat{a}(y)|dy \\
&= \sum_{u\in\mathbb{Z}^d}\int_{Q_u} |f_1(x-\theta_1 y)\cdots f_m(x-\theta_m y)K(y)\hat{a}(y)|dy \\
&\le \sum_{u\in\mathbb{Z}^d}\Big(\int_{Q_u} |f_1(x-\theta_1 y)\cdots f_m(x-\theta_m y)K(y)|^2 dy\Big)^{1/2}\Big(\int_{Q_u}|\hat{a}(y)|^2dy\Big)^{1/2} \\
&= \sum_{u\in\mathbb{Z}^d}\Big(\int_{Q_u} |f_1(x-\theta_1 y)\cdots f_m(x-\theta_m y)K(y)|^2 dy\Big)^{1/2}.
\end{aligned}$$

Therefore, we proved the claimed inequality (3.1).

To show inequality (1.7), we consider two cases:

Case 1: $2\le p<\infty$. Making use of Minkowski's inequality we obtain

$$\begin{aligned}
&\|T(f_1,\dots,f_m)\|_{L^p(\mathbb{R}^d)} \\
&\le \sum_{u\in\mathbb{Z}^d}\Big(\int_{\mathbb{R}^d}\Big(\int_{Q_u} |f_1(x-\theta_1 y)\cdots f_m(x-\theta_m y)K(y)|^2 dy\Big)^{p/2}dx\Big)^{1/p} \\
&\le \sum_{u\in\mathbb{Z}^d}\Big(\int_{Q_u}\Big(\int_{\mathbb{R}^d} |f_1(x-\theta_1 y)\cdots f_m(x-\theta_m y)K(y)|^p dx\Big)^{2/p}dy\Big)^{1/2} \\
&= \sum_{u\in\mathbb{Z}^d}\Big(\int_{Q_u}|K(y)|^2\Big(\int_{\mathbb{R}^d} |f_1(x-\theta_1 y)\cdots f_m(x-\theta_m y)|^p dx\Big)^{2/p}dy\Big)^{1/2} \\
&\le \sum_{u\in\mathbb{Z}^d}\Big\{\int_{Q_u}|K(y)|^2\Big[\Big(\int_{\mathbb{R}^d}|f_1|^{p\cdot\frac{p_1}{p}}dx\Big)^{\frac{p}{p_1}}\cdots\Big(\int_{\mathbb{R}^d}|f_m|^{p\cdot\frac{p_m}{p}}dx\Big)^{\frac{p}{p_m}}\Big]^{\frac{2}{p}}dy\Big\}^{\frac12} \\
&= B_2\prod_{i=1}^m\|f_i\|_{L^{p_i}(\mathbb{R}^d)}.
\end{aligned}$$

Case 2: $1\le p<2$. By Minkowski's inequality, we obtain

$$\begin{aligned}
&\|T(f_1,\dots,f_m)\|_{L^p(\mathbb{R}^d)} \\
&\le \Big(\int_{\mathbb{R}^d}\Big|\sum_{u\in\mathbb{Z}^d}\Big(\int_{Q_u} |f_1(x-\theta_1 y)\cdots f_m(x-\theta_m y)K(y)|^2 dy\Big)^{1/2}\Big|^p dx\Big)^{1/p} \\
&\le \sum_{u\in\mathbb{Z}^d}\Big(\int_{\mathbb{R}^d}\Big(\int_{Q_u} |f_1(x-\theta_1 y)\cdots f_m(x-\theta_m y)K(y)|^2 dy\Big)^{p/2}dx\Big)^{1/p} \\
&\le \sum_{u\in\mathbb{Z}^d}\Big\{\sum_{n\in\mathbb{Z}^d}\Big(\int_{P_n}\int_{Q_u} |f_1(x-\theta_1 y)\cdots f_m(x-\theta_m y)K(y)|^2 dydx\Big)^{p/2}\Big\}^{1/p},
\end{aligned}$$

where $P_n = n + [0,1)^d$ and we used Hölder's inequality for exponents $2/p$ and $(2/p)'$ over the space (P_n, dx) in the last inequality.

Let

$$A_{n,u} = \int_{P_n}\int_{Q_u} |f_1(x-\theta_1 y)\cdots f_m(x-\theta_m y)K(y)|^2 dydx$$

and set $f_i^{n,u} = f_i\chi_{P_n-\theta_i Q_u}$, $i = 1,2,\ldots,m$, $k_u = K\chi_{Q_u}$ and $f^{\vec{n,u}} = (f_i^{n,u})_{1\le i\le m}$. Using Theorem 1.3 (part (b)) with $q=1$, $r = p'/2$, and $p_i/2$ in place of p_i, we obtain

$$\begin{aligned}
A_{n,u} &\le \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} |f_1^{n,u}(x-\theta_1 y)\cdots f_m^{n,u}(x-\theta_m y)k_u(y)|^2 dydx\\
&= \int_{\mathbb{R}^d} |f^{\vec{n,u}}|^2 \otimes |k_u|^2(x)dx\\
&\le C\prod_{i=1}^m \||f_i^{n,u}|^2\|_{L^{p_i/2}(\mathbb{R}^d)}\||k_u|^2\|_{L^{p'/2}(\mathbb{R}^d)}\\
&= C\prod_{i=1}^m \|f_i^{n,u}\|^2_{L^{p_i}(\mathbb{R}^d)}\|k_u\|^2_{L^{p'}(\mathbb{R}^d)}.
\end{aligned}$$

Therefore, it follows from Hölder's inequality that

$$\begin{aligned}
&\|T(f_1,\ldots,f_m)\|_{L^p(\mathbb{R}^d)}\\
&\le C\sum_{u\in\mathbb{Z}^d}\Big(\sum_{n\in\mathbb{Z}^d}\Big(\prod_{i=1}^m \|f_i^{n,u}\|^2_{L^{p_i}(\mathbb{R}^d)}\|k_u\|^2_{L^{p'}(\mathbb{R}^d)}\Big)^{p/2}\Big)^{1/p}\\
&= C\sum_{u\in\mathbb{Z}^d}\|k_u\|_{L^{p'}(\mathbb{R}^d)}\Big(\sum_{n\in\mathbb{Z}^d}\Big(\int_{\mathbb{R}^d}|f_1^{n,u}|^{p_1}dx\Big)^{\frac{p}{p_1}}\cdots\Big(\int_{\mathbb{R}^d}|f_m^{n,u}|^{p_m}dx\Big)^{\frac{p}{p_m}}\Big)^{\frac{1}{p}}\\
&\le C\sum_{u\in\mathbb{Z}^d}\|K\chi_{Q_u}\|_{L^{p'}(\mathbb{R}^d)}\Big\{\prod_{j=1}^m\Big(\sum_{n\in\mathbb{Z}^d}\Big(\int_{P_n-\theta_j Q_u}|f_j|^{p_j}dx\Big)^{\frac{p}{p_j}\frac{p_j}{p}}\Big)^{\frac{p}{p_j}}\Big\}^{\frac{1}{p}}\\
&\le C\,B_p\prod_{i=1}^m\|f_i\|_{L^{p_i}(\mathbb{R}^d)}.
\end{aligned}$$

Thus, we complete the proof of Theorem 1.1.

□

References

[1] J. Bergh and J. Löfström, *Interpolation spaces. An Intorduction.* Grundlehren der mathematische Wissenschaften 223, Springer, Berlin-Heidelberg-New York, 1976.

[2] F. Bernicot, *L^p estimates for non smooth bilinear Littlewood-Paley square functions on $\mathbb{R}$*, Math. Ann. **351** (2011), no. 1, 1–49.

[3] C. Bennett, R. Sharpley, *Interpolation of Operators*, Academic Press, Boston, 1988.

[4] F. Bernicot, S. Shrivastava, *Bounedness of smooth bilinear square functions and applications to some bilinear pseudo-differential operators*, Indiana Univ. Math. J. **60** (2011), no. 1, 233–268.

[5] L. Grafakos and M. Mastyło, *Analytic families of multilinear operators*, Nonlinear Analysis, **107** (2014), 47–62.

[6] V. S. Guliyev, S. A. Nazirova, *O' Neil inequality for multilinear convolutions and some applications*, Int. Eq. Oper. Theory **60** (2008), no. 4, 485–497.
[7] M. Lacey, *On bilinear Littlewood-Paley square functions*, Publ. Mat. **40** (1996), no. 2, 387–396.
[8] M. Lacey, C. Thiele, *L^p estimates on the bilinear Hilbert transform for $2 < p < \infty$*, Ann. of Math. **146** (1997), no. 3, 693–724.
[9] M. Lacey, C. Thiele, *On Calderón's conjecture*, Ann. of Math. **149** (1999), no. 2, 475–496.
[10] P. Mohanty, S. Shrivastava, *A note on the bilinear Littlewood-Paley square function*, Proc. Amer. Math. Soc. **138** (2010), no. 6, 2095–2098.
[11] P. K. Ratnakumar, S. Shrivastava, *On bilinear Littlewood-Paley square functions*, Proc. Amer. Math. Soc. **140** (2012), no. 12, 4285–4293.
[12] P. K. Ratnakumar, S. Shrivastava, *A remark on bilinear Littlewood-Paley square functions*, Monatsh. Math. DOI 10.1007/s00605-014-0668-5.
[13] A. Zygmund, *Trigonometric Series*, Vol. II, 2nd ed., Cambridge University Press, Cambridge, UK, 1959.

Loukas Grafakos, Department of Mathematics, University of Missouri, Columbia, MO, 65211, USA
E-mail address: grafakosl@missouri.edu

She He, School of Mathematical Sciences, Beijing Normal University, Laboratory of Mathematics and Complex Systems, Ministry of Education, Beijing, 100875, P. R. China
E-mail address: amyhesha@163.com

Qingying Xue, School of Mathematical Sciences, Beijing Normal University, Laboratory of Mathematics and Complex Systems, Ministry of Education, Beijing 100875, P. R. China
E-mail address: qyxue@bnu.edu.cn